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LETTER TO THE EDITOR

An undular bore solution for the higher-order Korteweg–de Vries equation

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Abstract
Undular bores describe the evolution and smoothing out of an initial step in mean height and are frequently observed in both oceanographic and meteorological applications. The undular bore solution for the higher-order Korteweg–de Vries (KdV) equation is derived, using an asymptotic transformation which relates the KdV equation and its higher-order counterpart. The higher-order KdV equation considered includes all possible third-order correction terms (where the KdV equation retains second-order terms). The asymptotic transformation is then applied to the KdV undular bore solution to obtain the higher-order undular bore. Examples of higher-order undular bores, describing both surface and internal waves, are presented. Key properties, such as the amplitude and speed of the lead soliton and the width of the bore, are found. An excellent comparison is obtained between the analytical and numerical solutions. Also, it is illustrated how an asymptotic transformation and numerical solutions can be combined to generate hybrid asymptotic-numerical solutions, thus avoiding the severe instabilities associated with numerical schemes for the higher-order KdV equation.

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1. Introduction

Flow over topography is an important, widely occurring phenomenon in both oceanography and meteorology. Oceanographic applications include the generation of highly nonlinear internal waves on an evolving bore. For example, Holloway [1] describes the evolution of internal bores on the Australian North-West Shelf and Cummins *et al* [2] describes observations of the generation and propagation of internal waves in Knight Inlet, BC, Canada. A frequently occurring meteorological application is the morning glory, a series of roll clouds formed by the flow of breezes over a mountain range. Christie [3] has detailed observations and descriptions of the morning glory which forms over the Gulf of Carpentaria in Northern Australia.

The Korteweg–de Vries (KdV) equation arises as an approximate equation governing weakly nonlinear long waves when second-order nonlinear and dispersive terms are retained; see Whitham [4]. By retaining terms up to third order, the higher-order KdV equation

$$\eta_t + 6\eta\eta_x + \eta_{3x} + \alpha c_1\eta^2\eta_x + \alpha c_2\eta_x\eta_{xx} + \alpha c_3\eta\eta_{3x} + \alpha c_4\eta_{5x} = 0, \qquad \alpha \ll 1, \tag{1}$$

results, where α is a non-dimensional measure of the (small) wave amplitude. This equation describes the evolution of steeper waves of shorter wavelength than does the KdV equation. Marchant and Smyth [5] derived the version of (1) appropriate for surface waves on shallow water, in which case

$$c_1 = -1, \quad c_2 = \tfrac{23}{6}, \quad c_3 = \tfrac{5}{3}, \quad c_4 = \tfrac{19}{60}. \tag{2}$$

Whitham [4] developed modulation theory to study slowly varying wavetrains. In the case of the KdV equation these modulation equations were found to form a system of three first-order hyperbolic pdes for the properties of the modulated periodic cnoidal wave. A particular solution of these modulation equations is a centred simple wave; see Gurevich and Pitaevskii [6]. Physically the simple wave solution represents an undular bore which describes the evolution of an initial step in mean height. It was found to be in good agreement with numerical solutions of the KdV equation by Fornberg and Whitham [7].

The forced KdV equation has been widely used to describe the resonant flow of a stratified fluid over topography. This resonant flow consists of upstream and downstream flows which can be taken to be modulated unsteady wavetrains modelled using the KdV undular bore solution; see Smyth [8]. In physical applications, however, higher-order effects have been found to be important; see Melville and Helfrich [9] or Lamb and Yan [10].

Marchant and Smyth [5] used the asymptotic transformation

$$\eta = u + \frac{\alpha c_1}{12}(2u^2 + u_{xx}), \qquad \alpha \ll 1, \tag{3}$$

to transform (1) with only the higher-order nonlinear term c_1 (with the other c_i all zero) to the KdV equation. This transformation was used to derive modulation equations for this special case of (1). The higher-order undular bore solution was then used to model resonant flow over topography more accurately. Marchant [11] considered a non-local asymptotic transformation which allowed the higher-order KdV equation (1) to be transformed to the KdV equation.

The linearized dispersion relation for the higher-order KdV equation (1) is $\omega = -k^3 + \alpha c_4 k^5$. The linear wave speed is then not bounded for large wavenumber k. Hence when (1) is solved numerically, high-frequency waves of small amplitude, generated by the numerical discretization, are propagated at high speeds. This can cause numerical instabilities to develop and limits the numerical methods available for solving (1) to the case when α is small; see Marchant and Smyth [12] or Lamb and Yan [10].

In section 2 an alternate transformation to that of Marchant [11] is applied to reduce the higher-order KdV equation (1) to the KdV equation, from which the undular bore solution of the higher-order KdV equation is derived. In section 3 comparisons are made between the analytical and analytical–numerical solutions for two examples of higher-order undular bores. Hybrid analytical–numerical solutions of the higher-order KdV equation are developed in order to avoid the instabilities that occur in the numerical solution of (1).

2. The higher-order undular bore

Marchant [11] used an asymptotic transformation to relate the higher-order KdV equation (1) and the KdV equation and so obtained the two-soliton solution for the higher-order KdV equation (1). Here we use an alternate transformation to find a higher-order undular bore solution. If we substitute

$$\eta = u + \alpha c_5 u^2 + \alpha c_6 u_{xx} + \alpha c_7 u_x \int_{Ut}^{x} (u(p,t) - \beta)\,\mathrm{d}p,$$
$$\tau = t + \alpha \frac{c_4}{3} x, \qquad \xi = x + \alpha c_7 \beta (x - Ut) + \alpha c_7 D t, \qquad \alpha \ll 1, \qquad \text{where} \tag{4}$$
$$c_5 = \frac{1}{6}(c_3 - c_1 + 4c_4), \qquad c_6 = \frac{1}{12}(c_2 - 6c_4 - c_1), \qquad c_7 = \frac{1}{3}(8c_4 - c_3),$$
$$D = Uu - 3u^2 - u_{\xi\xi}$$

into the higher-order KdV equation (1), then $u(\xi, \tau)$ is a solution of the KdV equation $u_\tau + 6uu_\xi + u_{3\xi} = 0$ when terms of $\mathrm{O}(\alpha^2)$ are neglected. The transformation used by Marchant [11] was suitable for transforming soliton solutions only. Here the transformation needed is significantly different as it must be suitable for the transformation of the periodic cnoidal wave solution. In the transformation β and U are the mean level and phase velocity of the KdV cnoidal wave solution, respectively. It is also important that the transformation introduce no secular terms. This requirement is the reason for β appearing in the integrand of the non-local term in the transformation (with coefficient c_7), as the mean level of $u - \beta$ is zero. Also, the constant D is associated with the first integral of the KdV equation.

KdV modulation theory is based on the periodic cnoidal wave solution

$$u = \beta + \frac{2a}{m}[1 - m - f(m) + m\,\mathrm{cn}^2(K(m)\theta/\pi, m)],$$
$$\text{where} \quad f(m) = \frac{E(m)}{K(m)}, \qquad \theta = k\xi - \omega\tau, \tag{5}$$

and k is the wavenumber, ω the frequency, $U = \omega/k$ the phase speed, β the mean height and a the amplitude of the wavetrain. $K(m)$ and $E(m)$ are complete elliptic integrals of the first and second kinds, while m is the modulus squared of the elliptic integrals. In the limit $m \to 1$ (5) becomes the KdV soliton, while the limit $m \to 0$ represents linear, sinusoidal waves of small amplitude.

The simple wave solution of the KdV modulation equations was derived by Gurevich and Pitaevskii [6] and corresponds physically to an undular bore which links the level A behind the bore to the level B in front of the bore. It is

$$u = A - (A - B)m + 2(A - B)m\,\mathrm{cn}^2(K\theta/\pi, m),$$
$$a = 2(A - B)m, \qquad \beta = 2B - A + (A - B)(2f + m),$$
$$k = \pi K^{-1}(A - B)^{\frac{1}{2}}, \qquad U = 2A + 4B + 2(A - B)m,$$
$$p = A + (A - B)m, \qquad q = A - (A - B)m, \tag{6}$$
$$\text{on} \quad \frac{\xi}{\tau} = \lambda = U - 2(A - B)\frac{m(1 - m)}{f - (1 - m)},$$
$$\text{where} \quad 12B - 6A \leqslant \frac{\xi}{\tau} \leqslant 4A + 2B.$$

The periodic cnoidal wave (5) has constant amplitude, mean height, wavenumber and modulus squared m. By contrast, (6) is a cnoidal wavetrain for which the properties are modulated, or slowly varying. Within the bore the modulus squared m varies, from $m = 1$ at the leading edge to $m = 0$ at the trailing edge. At the leading edge solitons of amplitude $2(A - B)$ on a mean level of B occur, while at the trailing edge there are sinusoidal waves of small amplitude on a mean height A. The width of the bore is $10(A - B)\tau$. Hence undular bores only occur if there is a step down in mean height, i.e. where $A > B$. The quantities p and q are the peak and trough heights of the wave and represent the envelopes of the wavetrain.

Substituting the expression for β within (6) into the transformation (4) gives

$$\begin{aligned}\beta = 2B - A + (A-B)(2f+m) + \frac{\alpha}{3}c_5[(A-B)^2(2-5m+3m^2\\ +(4m-2)f) + 6B^2 - 3A^2 + (A^2-B^2)(6f+3m)]\\ +\alpha\frac{4}{3}c_7(A-B)^2[3(1-f)^2 - 2(1-f)(1+m)+m],\end{aligned} \tag{7}$$

as the mean height of the higher-order undular bore. The new mean levels at the front and rear of the bore are then

$$\begin{aligned}\beta &= A + \alpha c_5 A^2, \qquad \text{as} \quad m \to 0,\\ \beta &= B + \alpha c_5 B^2, \qquad \text{as} \quad m \to 1.\end{aligned} \tag{8}$$

Hence the transformation has changed the mean levels at the front and rear of the undular bore. To change the mean levels at the two ends of the bore back to the original values we rescale the boundary levels by

$$A = A^* - \alpha c_5 A^{*2}, \qquad B = B^* - \alpha c_5 B^{*2}, \tag{9}$$

which gives the mean height of the undular bore as

$$\begin{aligned}\beta = 2B - A + (A-B)(2f(m)+m) + \frac{\alpha}{3}c_5(A-B)^2[2-5m+3m^2\\ +(4m-2)f] + \alpha\frac{4}{3}c_7(A-B)^2[3(1-f)^2 - 2(1-f)(1+m)+m],\end{aligned} \tag{10}$$

where we have dropped the stars. Similar use of the transformation and the scaling (9) allows expressions for the higher-order amplitude and peak and trough heights of the bore to be found.

Let us now consider the effect of the transformation (4) on the space and time variables ξ and τ. We find that

$$\begin{aligned}\frac{x}{t} &= \lambda - \alpha\lambda_1 + \frac{\alpha}{3}c_4\lambda^2 - \alpha c_7 D - \alpha c_7[2B - A + (A-B)m\\ &\quad + 2(A-B)f(m)](\lambda - U), \qquad \text{where}\\ \lambda_1 &= 2A^2 + 4B + 2(A^2-B^2)m - 2(A^2-B^2)\frac{m(1-m)}{f-(1-m)},\\ D &= -(A-B)^2m^2 + 2A(A-B)m - A^2 + 4AB,\end{aligned} \tag{11}$$

where the characteristic and phase speeds, λ and U, from the KdV bore (6), have been used. Also the scalings (9) have been applied. Again, the transformation can be applied in a similar manner to obtain the higher-order phase velocity and wavenumber.

Summarizing, the higher-order KdV undular bore solution is

$$\begin{aligned}\beta &= 2B - A + (A-B)(2f(m)+m) + \frac{\alpha}{3}c_5(A-B)^2[2-5m+3m^2+(4m-2)f]\\ &\quad +\alpha\frac{4}{3}c_7(A-B)^2[3(1-f)^2 - 2(1-f)(1+m)+m],\\ a &= (A-B)m + \alpha(c_5 m + 2c_6(m^2-2m))(A-B)^2,\\ p &= A + (A-B)m + \alpha c_5[(A-B)^2m^2 + 2A(A-B)m\\ &\quad - (A^2-B^2)m] - \alpha c_6(A-B)^2 4m,\\ q &= A - (A-B)m + \alpha c_5[(A-B)^2m^2 - 2A(A-B)m\\ &\quad + (A^2-B^2)m] + \alpha c_6(A-B)^2 4m(1-m),\end{aligned}$$

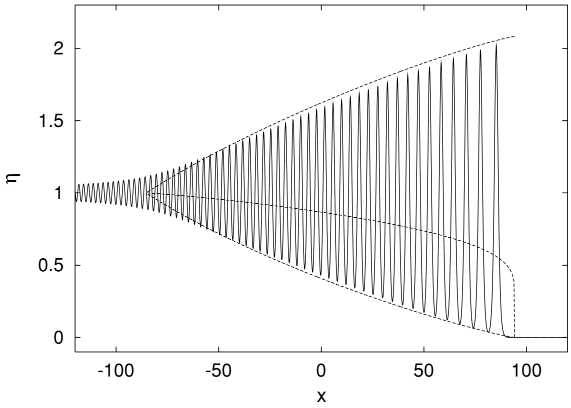

Figure 1. The elevation η versus x at $t = 25$ for $A = 1$ and $B = 0$. The c_i are given by (2) and $\alpha = 0.25$. Shown are the analytical higher-order KdV bore (dashed lines) and analytical–numerical (solid lines) solution of (1). The analytical solution consists of the wave peak and trough envelopes and the mean height.

$$U = C - \alpha c_7 D + \alpha \frac{c_4}{3} C^2 - \alpha c_5[6B^2 + 2(A^2 - B^2)(1 + m)]$$
$$k = \pi K^{-1}(A - B)^{\frac{1}{2}} \left[1 + \alpha c_7 D - \alpha \frac{c_4}{3} C - \frac{1}{2}\alpha c_5 (A + B)\right],$$
$$C = 2A + 4B + 2(A - B)m. \tag{12}$$

In the limit $m \to 1$ we have $a = 2(A - B) + \alpha 2(c_5 - 2c_6)(A - B)^2$ as the amplitude of the lead soliton in the higher-order bore. Also, taking the characteristic (11) in the limits $m \to 0$ and $m \to 1$ gives the extent of the bore as

$$12B - 6A + \alpha A^2(12c_4 + 6c_5 - c_7) + \alpha AB\left(4c_7 - \frac{144}{3}c_4\right) + \alpha B^2\left(\frac{144}{3}c_4 - 12c_5\right) < \frac{x}{t} < 2B + 4A + \alpha A^2\left(\frac{16}{3}c_4 - 4c_5\right) + \alpha AB\left(\frac{16}{3}c_4 + 4c_7\right) + \alpha B^2\left(\frac{4}{3}c_4 - 2c_5 - c_7\right). \tag{13}$$

The leading edge of the bore is determined by the velocity of the lead soliton, while the trailing edge is determined by the group velocity of the linear radiation. The width of the bore at unit time is

$$10(A - B) + \alpha A^2\left(c_7 - \frac{20}{3}c_4 - 10c_5\right) + \alpha\frac{160}{3}c_4 AB - \alpha B^2\left(\frac{140}{3}c_4 - 10c_5 + c_7\right), \tag{14}$$

and the condition for the bore to exist remains $A > B$.

3. Results and discussion

Here the approximate analytical solution for the higher-order undular bore will be compared with hybrid analytical–numerical solutions of the higher-order KdV equation (1). The numerical solutions were found using a hybrid Runge–Kutta finite-difference scheme; see Marchant and Smyth [12]. Figure 1 shows the elevation η versus x at $t = 25$ for $A = 1$ and $B = 0$. The other parameters are $\alpha = 0.25$ and the c_i all given by (2). Hence this

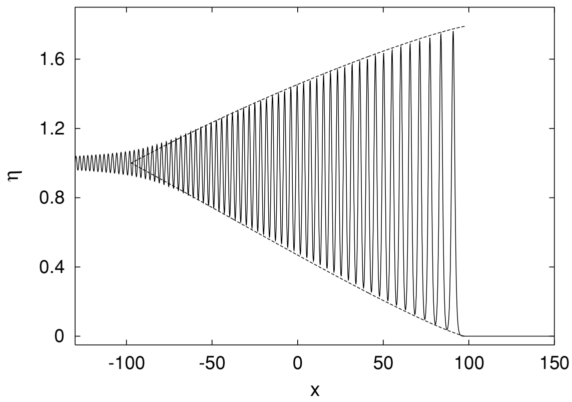

Figure 2. The elevation η versus x at $t = 25$ for $A = 1$ and $B = 0$. The other parameters are given by (16). Shown are the analytical higher-order KdV bore (dashed lines) and numerical (solid lines) solutions of (1). The analytical solution consists of the wave peak and trough envelopes and the mean height.

example describes the evolution of a higher-order bore for surface waves on shallow water. The comparison between the analytical and the numerical–analytical solution is excellent. The analytical solution consists of an undular bore which lies in the region $-85.4 < x < 94.2$ and takes the solution from a mean level of $\eta = 1$ at $x = -85.4$ to $\eta = 0$ at $x = 94.2$. The analytical prediction, from (12), for the lead soliton amplitude is 2.08, compared with the numerical amplitude of 2.02. This is some 4% higher than the lead soliton in a KdV undular bore, which has an amplitude of 2. Moreover, a KdV bore propagates between $-150 < x < 100$, so the lead soliton travels 6% slower in the higher-order bore and the width of the higher-order bore is 28% less.

The calculation of numerical solutions for the higher-order KdV equation is limited to small values of α due to numerical instabilities. It is found that the higher-order terms $\eta^2\eta_x$ and $\eta\eta_{3x}$ do not affect the stability of the numerical scheme due to their qualitative similarity to the nonlinear and dispersive terms ($\eta\eta_x$ and η_{3x}, respectively) of the KdV equation. However, the inclusion of the higher-order terms $\eta_x\eta_{xx}$ and η_{5x} leads to severe limitations on the numerical stability. To overcome this numerical instability the hybrid technique

$$\begin{aligned} &u_\tau + 6uu_\xi + u_{3\xi} + \alpha(c_1 - 12c_4)u^2u_\xi + \alpha(c_3 - 8c_4)uu_{3\xi} = 0, \\ &\eta = u + \frac{\alpha}{12}(c_2 - 18c_4)u_{xx}, \qquad \tau = t + \alpha\frac{c_4}{3}x, \end{aligned} \tag{15}$$

is used. The first of (15) is solved numerically and then the second of (15) is used to transform the numerical solution. This gives a hybrid numerical–analytical solution of (1). This hybrid solution only differs from a direct solution of (1) at $O(\alpha^2)$, which is consistent with the magnitude of the neglected terms in the derivation of (1). The transformation in (15) is local, does not change the mean level and is straightforward to apply to any numerical solution. Figure 2 shows the elevation η versus x at $t = 25$ for $A = 1$ and $B = 0$. The other parameters are chosen as

$$\alpha c_1 = -0.191, \quad \alpha c_2 = 1.39, \quad \alpha c_3 = 0.186, \quad \alpha c_4 = 0.0573. \tag{16}$$

Figure 2 shows a rescaled version of undular bore evolution for internal waves for the example shown in figure 8 of Lamb and Yan [10]. They showed the bore at time $t = 5.7$. Here we show its evolution at a later time, $t = 25$. An excellent comparison between the analytical and numerical solutions is obtained. The analytical solution consists of an undular bore which lies in the region $-97.2 < x < 97.5$ and takes the solution from a mean level of $\eta = 1$ at $x = -97.2$ to $\eta = 0$ at $x = 97.5$. The analytical lead soliton amplitude is 1.79, compared with the numerical amplitude 1.76. This is some 11% lower than the KdV lead soliton amplitude. Moreover, a KdV bore propagates between $-150 < x < 100$, so the lead soliton travels 3% slower in the higher-order bore, with the width of the higher-order bore 22% less.

Figure 8 of Lamb and Yan [10] compared numerical results for internal wave undular bore evolution for the KdV and higher-order KdV equations and also for the full governing equations, which comprise the internal water wave equations for incompressible, inviscid flow. They found that there was little difference between numerical solutions of (1) and the internal water wave equations. Equation (1) was found to give accurate predictions for the lead soliton amplitude (within 2%) and the width of the bore. They also found that the KdV equation was not very accurate, as it substantially over predicts the lead soliton amplitude and bore width, as given by the internal water wave equations (see discussion above).

In summary, the analytical higher-order bore solution (12) is found to accurately describe undular bore evolution as given by the higher-order KdV equation. While it was directly shown that it compares well with numerical solutions of (1), Lamb and Yan [10] showed that the difference between numerical predictions of (1) and of the internal water wave equations is small for a realistic example of undular bore evolution for internal waves.

4. Conclusion

An analytical solution (12) for a higher-order KdV bore was developed using an asymptotic transformation. It was found that the comparison between analytical and numerical solutions is excellent. Moreover, analytical expressions were found for key quantities in the higher-order bore, such as the amplitude and speed of the lead soliton and the width of the bore. The higher-order KdV bore solution may be useful in understanding experimental and numerical results from important applications in oceanography and meteorology. Also, a practical and simple technique to obtain hybrid solutions of the higher-order KdV equation was presented. These solutions are very difficult to obtain directly, due to numerical instability.

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